Section 12.4 The Cross Product

DEF: The **determinant** of a 2×2 matrix is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Ex1. Calculate $\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix}$

$(2)(4) - (-6)(1) = 14$

$8 \quad + 6 \quad = 14$

DEF: The **determinant** of a 3×3 matrix is defined by

add minus sign

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$

DEF: Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$. The **cross product** of $\vec{a}$ and $\vec{b}$ is a **vector** and is defined as

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

i.e.

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

been swapped due to neg sign

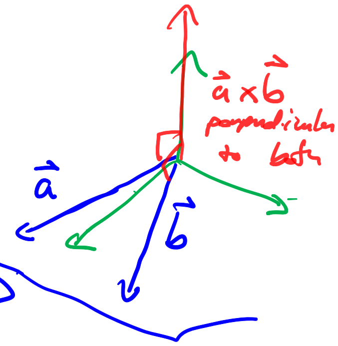
Ex2. Let $\vec{a} = \langle 1, 3, 4 \rangle$ and $\vec{b} = \langle 2, 6, 8 \rangle$. Calculate $\vec{a} \times \vec{b}$.

add minus sign

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 6 & 8 \end{vmatrix} = \langle \underbrace{(3)(8) - (4)(6)}_{0}, -\underbrace{((1)(8) - (2)(4))}_{0}, \underbrace{(1)(6) - (2)(3)}_{0} \rangle$$

$= \langle 0, 0, 0 \rangle$ "The zero vector"

Ex3. (Key Fact) Prove that the vector $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.



Let $\vec a = \langle a_1, a_2, a_3 \rangle$ and $\vec b = \langle b_1, b_2, b_3 \rangle$

$$\vec a \times \vec b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2 b_3 - a_3 b_2, -(a_1 b_3 - a_3 b_1), a_1 b_2 - a_2 b_1 \rangle$$

Now, $(\vec a \times \vec b) \cdot \vec a = \cancel{a_1 a_2 b_3} - \cancel{a_1 a_3 b_2} - \cancel{a_1 b_3 a_2} + \cancel{a_3 b_1 a_2} + \cancel{a_1 b_2 a_3} - \cancel{a_2 b_1 a_3}$

$= 0$ (number!)

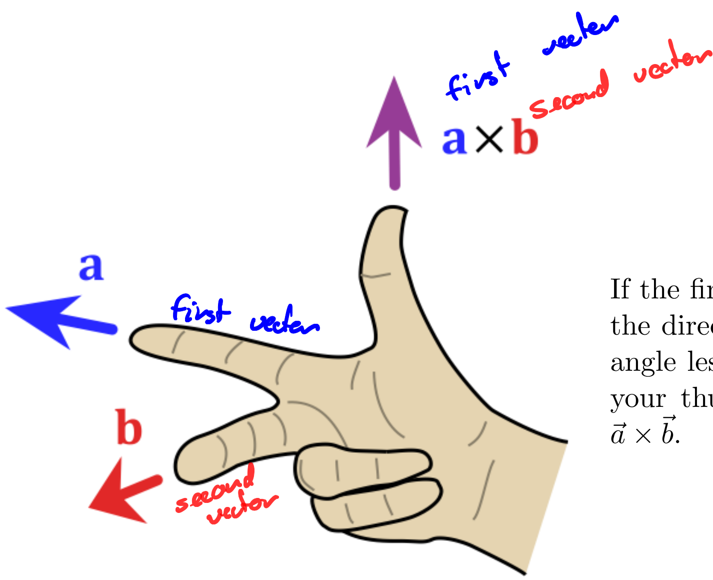
so $(\vec a \times \vec b) \cdot \vec a$ are perpendicular

similarly, $(\vec a \times \vec b) \cdot \vec b = 0 \rightarrow \vec a \times \vec b \perp \vec b$

means perpendicular

Rmk: $\vec{a} \times \vec{b}$ points in the direction perpendicular to the plane through $\vec{a}$ and $\vec{b}$.

Ex4. The direction of $\vec{a} \times \vec{b}$ is given the by the right-hand rule.



If the fingers of your right hand curl in the direction of a rotation (through an angle less than 180°) from $\vec{a}$ to $\vec{b}$, then your thumb points in the direction of $\vec{a} \times \vec{b}$.

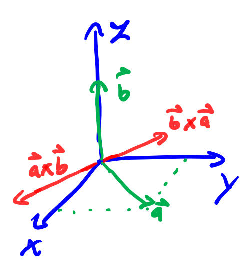
Let $\vec{a} = \langle 1, 1, 0 \rangle$ and $\vec{b} = \mathbf{k}$. Calculate $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$. Sketch $\vec{a}$, $\vec{b}$, $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$.

$\vec b = \langle 0, 0, 1 \rangle$

$$\vec a \times \vec b = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 1, -(1), 0 \rangle = \langle 1, -1, 0 \rangle$$

$$\vec b \times \vec a = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \langle -1, -(-1), 0 \rangle = \langle -1, 1, 0 \rangle$$

Note: $\vec a \times \vec b = -(\vec b \times \vec a)$



Theorem. Algebraic Properties of the Cross Product:
Let $\vec{a}$ and $\vec{b}$ be three dimensional vectors and let λ be a real number, then:

(1) $(\lambda\vec{a}) \times \vec{b} = \lambda(\vec{a} \times \vec{b})$

(2) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(3) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

(4) $\vec{0} \times \vec{a} = \vec{0}$

Proof of (1): Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$. Let λ be a real number

Left hand side (LHS):

$$(\lambda \vec{a}) \times \vec{b} = \begin{vmatrix} i & j & k \\ \lambda a_1 & \lambda a_2 & \lambda a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle \lambda a_2 b_3 - \lambda a_3 b_2, -(\lambda a_1 b_3 - \lambda a_3 b_1), \lambda a_1 b_2 - \lambda a_2 b_1 \rangle$$

Right Hand Side (RHS):

$$\lambda(a \times b) = \lambda \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \lambda \langle a_2 b_3 - a_3 b_2, -(a_1 b_3 - a_3 b_1), a_1 b_2 - a_2 b_1 \rangle$$
$$= \langle \lambda a_2 b_3 - \lambda a_3 b_2, -(\lambda a_1 b_3 - \lambda a_3 b_1), \lambda a_1 b_2 - \lambda a_2 b_1 \rangle$$

LHS = RHS

$(\lambda a) \times b = \lambda (a \times b)$

Lemma: Let $\vec{a}$ and $\vec{b}$ be three dimensional vectors. Then $\overbrace{||\vec{a} \times \vec{b}||^2}^{\text{LHS}} = \overbrace{||\vec{a}||^2 ||\vec{b}||^2 - (\vec{a} \cdot \vec{b})^2}^{\text{RHS}}$.

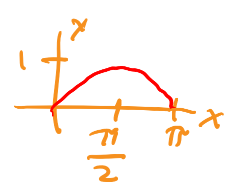
Theorem: If θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$, then $||\vec{a} \times \vec{b}|| = ||\vec{a}|| \, ||\vec{b}|| \, \sin\theta$.
Proof:

From the Lemma: $||\vec{a} \times \vec{b}||^2 = ||\vec{a}||^2 ||\vec{b}||^2 - (\vec{a} \cdot \vec{b})^2$

$$= ||\vec{a}||^2 \, ||\vec{b}||^2 - (||\vec{a}|| \, ||\vec{b}|| \cos\theta)^2$$
$$= ||a||^2 ||b||^2 - ||a||^2 ||b||^2 \cos^2\theta$$
$$= ||a||^2 ||b||^2 (1 - \cos^2\theta)$$
$$= ||a||^2 ||b||^2 (\sin^2\theta)$$

$||a \times b|| = ||a|| \, ||b|| \, |\sin\theta| \qquad (0 \le \theta \le \pi)$

$||a \times b|| = ||a|| \, ||b|| \, \sin\theta$

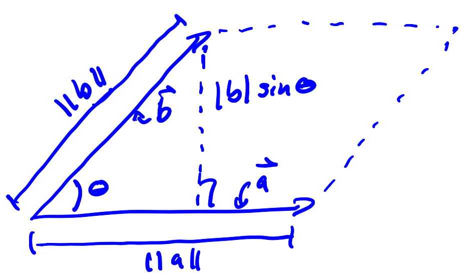


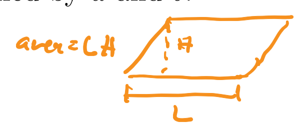
Ex5. Prove that two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.

Let θ be the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$

$$\vec{a} \parallel \vec{b} \iff \theta = 0 \text{ or } \theta = \pi \iff \sin\theta = 0$$
$$\iff |a||b| \sin\theta = 0$$
$$\iff |a \times b| = 0$$
$$\iff a \times b = \vec{0}$$

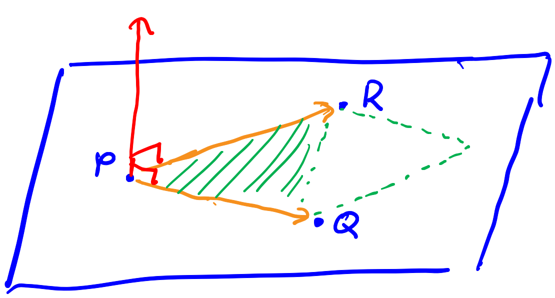
Ex6. Find a cross product formula for the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.





$$\text{Area} = \underbrace{|a||b| \sin\theta}$$
$$\text{Area} = \|a \times b\|$$

Ex7. Find a nonzero vector perpendicular to the plane that contains the points $P = (1, 0, 1)$, $Q = (-2, 1, 3)$, and $R = (4, 2, 5)$. Find the area of the triangle PQR.



$$\vec{PQ} = \langle -3, 1, 2 \rangle$$
$$\vec{PR} = \langle 3, 2, 4 \rangle$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \boxed{\langle 0, 18, -9 \rangle}$$

$-12-6$

$-6-3$

One nonzero vector perpendicular to the plane is $\langle 0, 18, -9 \rangle$

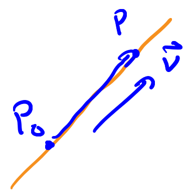
area of triangle $= \frac{1}{2}$ (area of parallelogram)

$$= \frac{1}{2} \|\vec{PQ} \times \vec{PR}\| = \frac{1}{2} \|\langle 0, 18, -9 \rangle\|$$
$$= \frac{1}{2} \|9 \langle 0, 2, -1 \rangle\| = \frac{9}{2}\sqrt{0+4+1}$$
$$= \frac{9\sqrt{5}}{2}$$

$y = mx + b$ doesn't work in $\mathbb{R}^3$ $\qquad \begin{cases} x = t \\ y = mt + b \end{cases}$

Section 12.5 Equations of Lines and Planes

Suppose that a line L in space passes through the point $P_0 = (x_0, y_0, z_0)$ and is parallel to the nonzero vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Let $P = (x, y, z)$ be an arbitray point on the line L. Then geometrically one sees that the vectors $\vec{v}$ and $\overrightarrow{P_0P}$ must be parallel.



$\overrightarrow{P_0P} \underset{\text{parallel}}{/\!/} \vec{v}$ $\qquad \overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$

so, $\langle x - x_0, y - y_0, z - z_0 \rangle = t \langle v_1, v_2, v_3 \rangle$

Thus, the line L is the set of all points $P = (x, y, z)$ for which $\quad \overrightarrow{P_0P} = t\vec{v} \quad$ for some number t. Now we can derive **parametric equations** of the line L. That is:

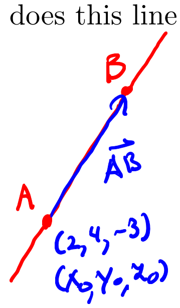
$x - x_0 = tv_1$ and $y - y_0 = tv_2$ and $z - z_0 = tv_3$

then $\boxed{x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3}$ "Parametric equations of the line"

We also write $(x(t), y(t), z(t)) = (x_0 + tv_1, y_0 + tv_2, z_0 + tv_3)$

In vector form $r(t) = \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle$

Ex1. Find parametric equations for the line that passes through the points $A = (2, 4 - 3)$ and $B = (3, -1, 1)$. Is the point $C = (4, -6, 5)$ on the line? How about $D = (0, 14, -10)$? At what point does this line intersect the xy-plane?



$\overrightarrow{AB} /\!/$ line, $\overrightarrow{AB} = \langle 1, -5, 4 \rangle$

Parametric equations of the line are:

$\boxed{x = 2 + t(1), \quad y = 4 + t(-5), \quad z = -3 + t(4)}$

Is C on the line?

we want $\underbrace{4 = 2 + t}_{t=2}, \underbrace{-6 = 4 - 5t}_{t=2}, \underbrace{5 = -3 + 4t}_{t=2}$

when $t = 2$, $C = (4, -6, 5)$ verifies the parametric equations of the line, so C is on the line.

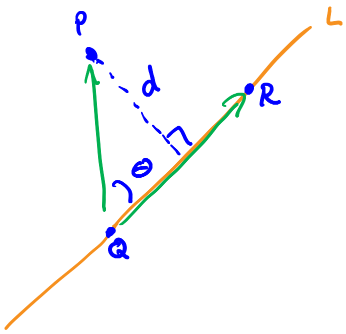
Is D on the line?

We want $\underbrace{0 = 2 + t}_{t=-2}, \underbrace{14 = 4 - 5t}_{t=-2}, \underbrace{-10 = -3 + 4t}_{t=-7/4}$

so D is <u>not</u> on the line.

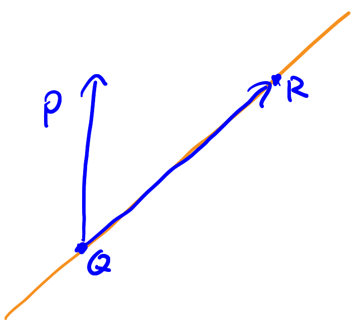
Recall: xy-plane means $z = 0$. Suppose the intersection point is $(x_1, y_1, 0)$. Then $x_1 = 2 + t$, $y_1 = 4 - 5t$, $0 = -3 + 4t$. Here $t = 3/4$, so $x_1 = 2 + \frac{3}{4}$, $y_1 = 4 - 5\left(\frac{3}{4}\right)$ the intersection point is $(x_1, y_1, 0) = \left(\frac{11}{4}, \frac{1}{4}, 0\right)$

Ex2. Let P be a point not on the line L that passes through the points Q and R. Derive a cross product formula for the distance d from the point P to the line L.



$$d = \|\overrightarrow{QP}\| \sin\theta$$
$$= \cancel{\|\overrightarrow{QP}\|}\ \frac{\|\overrightarrow{QP} \times \overrightarrow{QR}\|}{\cancel{\|\overrightarrow{QP}\|}\ \|\overrightarrow{QR}\|}$$
$$= \frac{\|\overrightarrow{QP} \times \overrightarrow{QR}\|}{\|\overrightarrow{QR}\|}$$

• Use the formula to find the distance from the point $P = (1, 1, 1)$ to the line through $Q = (0, 6, 8)$ and $R = (-1, 4, 7)$.



$$\overrightarrow{QP} = \langle 1, -5, -7 \rangle$$
$$\overrightarrow{QR} = \langle -1, -2, -1 \rangle$$

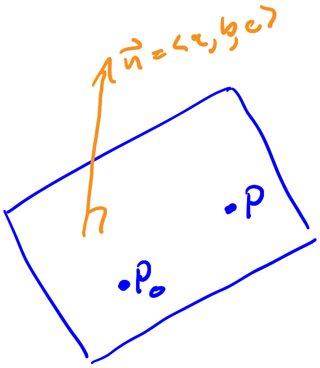
$$\overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} i & j & k \\ 1 & -5 & -7 \\ -1 & -2 & -7 \end{vmatrix} = \langle -9, 8, -7 \rangle$$

$$d = \frac{\|\overrightarrow{QP} \times \overrightarrow{QR}\|}{\|\overrightarrow{QR}\|} = \frac{\|\langle -9, 8, -7 \rangle\|}{\|\langle -1, -2, -1 \rangle\|} = \frac{\sqrt{81 + 64 + 49}}{\sqrt{1 + 4 + 1}} = \frac{\sqrt{194}}{\sqrt{6}} = \frac{\sqrt{97}}{\sqrt{3}}$$

The Point-Normal Equation for a Plane

Suppose that a plane S passes through the point $P_0 = (x_0, y_0, z_0)$ and is perpendicular (or normal) to the nonzero vector $\vec{n} = \langle a, b, c\rangle$. Sketch a generic point on the plane $P = (x, y, z)$ and deduce that the point $P = (x, y, z)$ must verify an equation of the form

$$ax + by + cz = d.$$



$\overrightarrow{P_0P} \cdot \vec{n} = 0$

$\langle x - x_0, y - y_0, z - z_0\rangle \cdot \langle a, b, c\rangle = 0$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$ax - ax_0 + by - by_0 + cz - cz_0 = 0$

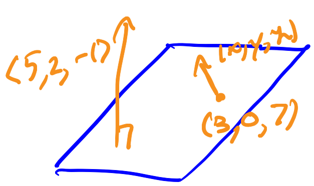
$ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}$

set $d = ax_0 + by_0 + cz_0$

so $\boxed{ax + by + cz = d}$ Eq. of the plane

(also called "Linear eqn. in x, y, z)

Ex3. Find an equation for the plane through the point $P_0 = (-3, 0, 7)$ perpendicular to the vector $\vec{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

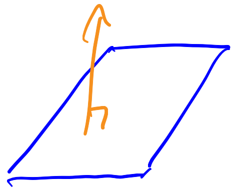


$\langle x + 3, y, z - 7\rangle \cdot \langle 5, 2, -1\rangle = 0$

$5(x + 3) + 2(y) + (-1)(z - 7) = 0$

$\boxed{5x + 2y - z = -22}$

Ex4. Find a nonzero vector normal to the plane given by $3x - 6y - 2z = 3$.

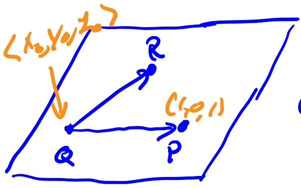


The equation of the plane is $ax + by + cz = d$ where a vector perpendicular to the plane is $\langle a, b, c\rangle$.

So, in this problem one vector normal to the plane is $\langle 3, -6, -2\rangle$

(Ex 7, Section 12.4)

Ex5. Find an equation for the plane containing the points $P = (1, 0, 1)$, $Q = (-2, 1, 3)$, and $R = (4, 2, 5)$.



$\vec{QP} = \langle 3, -1, -2 \rangle$

$\vec{QR} = \langle 6, 1, 2 \rangle$

$\vec{QP} \times \vec{QR} = \langle 0, -18, 9 \rangle = 9\langle 0, -2, 1 \rangle$

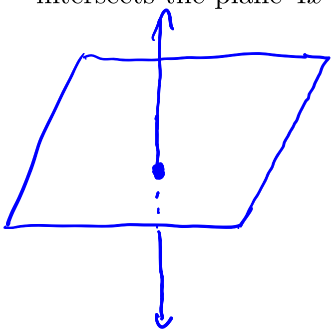
then, $\langle x-1, y-0, z-1 \rangle \cdot \langle 0, -2, 1 \rangle = 0$

$0(x-1) - 2(y-0) + 1(z-1) = 0$

$-2y + z - 1 = 0$

$-2y + z = 1$

Ex6. Find the point at which the line with parametric equations $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.



$4(2+3t) + 5(-4t) - 2(5+t) = 18$ (For some t)

$8 + 12t - 20t - 10 - 2t = 18$

$-16t - 2 = 18$

$-10t = 20 \rightarrow \boxed{t = -2}$

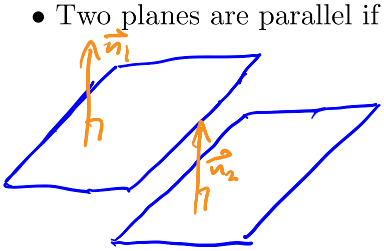
then the intersection point is

$(x, y, z) = (2+3(-2), -4(-2), 5+(-2))$

$\boxed{(x, y, z) = (-4, 8, 3)}$

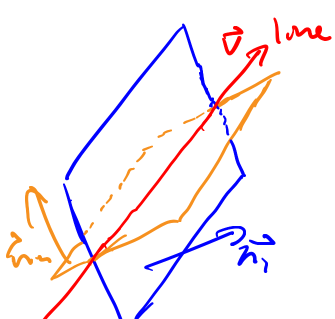
DEF:

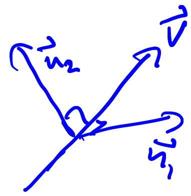
- Two planes are parallel if their normal vectors are parallel.



$\vec{n}_1 \parallel \vec{n}_2$

- Two planes that are not parallel intersect in a line.



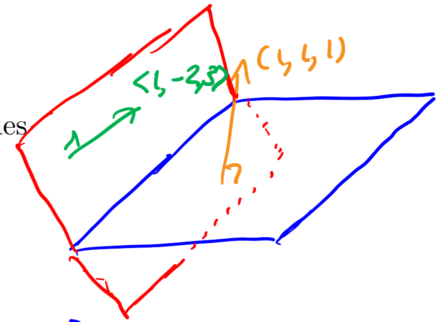


Consider:

$\vec{v} \parallel$ line, $\vec{v} \perp \vec{n}_1$ and $\vec{v} \perp \vec{n}_2$

Consider $\vec{v} = \vec{n}_1 \times \vec{n}_2$

Ex7. Find parametric equations for the line of intersection of the planes



$$x + y + z = 1, \qquad x - 2y + 3z = 1$$

$$\vec{v} = \langle 1,1,1\rangle \times \langle 1,-2,3\rangle = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$\vec{V} = \langle 5, -2, -3\rangle$ ($\vec{v}$ // line)

Let's get one point on the line

when $z=0$ $\begin{cases} x+y=1 \\ x+2y=1 \end{cases}$ -

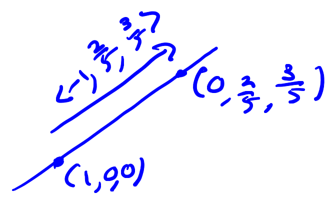
$3y=0 \Rightarrow y=0, x=1$

when $x=0$ $\begin{cases} y+z=1 \\ -2y+3z=1 \end{cases} \Rightarrow \begin{cases} 2y+2z=2 \\ -2y+3z=1 \end{cases}$

$5z=3$

$y = 1 - 3/5$

one point on the line is $(1,0,0)$

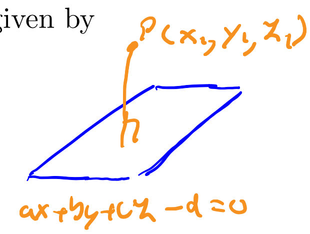


Parametric equations of the line are

$$\begin{cases} x = 1 + t(5) \\ y = 0 + t(-2) \\ z = 0 + t(-3) \end{cases}$$

Theorem [Distance from a Point to a Plane]
The distance D from the point $P = (x_1, y_1, z_1)$ to the plane $ax + by + cz = d$ is given by

$$D = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$



Proof: See textbook, page 829. This is a reading homework!

Ex8. Use the above formula to find the distance from the point $P = (1, -2, 4)$ to the plane

$$\underbrace{3x + 2y + 6z = 5.}_{3x+2y+6z-5=0}$$

$$D = \frac{|3(1)+2(-2)+6(4)-5|}{\sqrt{9+4+36}} = \frac{|3-4+24-5|}{\sqrt{49}} = \frac{18}{7}$$